Euler's Theorem.

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Q:- Given $z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$ Then Prove that $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z.$

Sol: Here $z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$

let $u = x^n f_1\left(\frac{y}{x}\right)$ and $v = y^{-n} f_2\left(\frac{x}{y}\right)$

i.e $\boxed{z = u + v}$

Here $u = x^n f_1\left(\frac{y}{x}\right)$ is a homogenous function in variable x and y with degree n.

Then by Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{——— (1)}$$

and $v = y^{-n} f_2\left(\frac{x}{y}\right)$ is also a homogenous function in variable x and y with degree $(-n)$.

Then by Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -nv \qquad \text{——— (2)}$$

Equation (1) partially diff w.r.t x and y

$$x\frac{\partial^2 u}{\quad} + \frac{\partial u}{\quad} + y\frac{\partial^2 u}{\quad} \qquad \text{——— (3)} \times \cancel{x}$$

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \quad \text{———} \ (3)\times x$$

and $$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y} \quad \text{———} \ (4)\times y$$

from (3) $\times x$ + (4) $\times y$, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx\frac{\partial u}{\partial x}+ny\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

——— (5)

Now partially diff w.r.t. x and y of equation (2), we get

$$x\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}+y\frac{\partial^2 v}{\partial x\partial y}=-n\frac{\partial v}{\partial x} \quad \text{———} \ (6)\times x$$

and $$x\frac{\partial^2 v}{\partial y\partial x}+y\frac{\partial^2 v}{\partial y^2}+\frac{\partial v}{\partial y}=-n\frac{\partial v}{\partial y} \quad \text{———} \ (7)\times y$$

from (6) $\times x$ + (7) $\times y$

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}+\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=-n\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)$$

——— (8)

Add Equation (5) and (8)

$$x^2\left(\frac{\partial^2}{\partial x^2}(u+v)\right)+2xy\frac{\partial^2}{\partial x\partial y}(u+v)+y^2\left(\frac{\partial^2}{\partial y^2}(u+v)\right)+x\frac{\partial(u+v)}{\partial x}+y\frac{\partial}{\partial y}(u+v)$$

$$=n\underbrace{\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)}_{}+$$

from (1)

from 2

$$= n\underbrace{\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)}$$

$$(-n)\underbrace{\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)}$$

from

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} =$$

$$n(nu) - n(-nv)$$

$$= n^2u + n^2v$$

$$= n^2(u+v)$$

$$= n^2 z$$

Hence $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$ Proved.

Q:- If $u = \tan^{-1}(y^2/x)$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$$

Sol:- Given $u = \tan^{-1}(y^2/x)$

$$\tan u = y^2/x = x\left(\frac{y^2}{x^2}\right) = x^1\left(\frac{y}{x}\right)^2 = z \text{ (say)}$$

i.e $z = \dfrac{y^2}{x} = x^1\left(\frac{y}{x}\right)^2 = x^1\phi\left(\frac{y}{x}\right)$ type.

$\therefore$ z is a homogenous function in two variables x and y with degree $= 1$.

Then By Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (1)z$$

$$x\frac{\partial(\tan u)}{\partial x}+y\frac{\partial(\tan u)}{\partial y}=\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\tan u}{\sec^2 u}=\frac{\sin u}{\cos u}\cdot\cos^2 u=\sin u\cos u$$

$$=\frac{1}{2}(2\sin u\cos u)$$

$$=\frac{1}{2}\sin 2u$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u \quad\text{———}(1)$$

Partially diff w.r.t x of Equation (1), we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\cos 2u\left(2\frac{\partial u}{\partial x}\right)$$

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\cos 2u\left(\frac{\partial u}{\partial x}\right)$$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\cos 2u\left(\frac{\partial u}{\partial x}\right)-\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}(\cos 2u-1)$$

Multiplying both sides by x, we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=x\frac{\partial u}{\partial x}(\cos 2u-1)\quad\text{———}(2)$$

Now,

Now,

Partially diff w.r.t y of Equation (1), we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=\cos 2u\,\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=\cos 2u\,\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial y}(\cos 2u-1)$$

Multiplying both sides by y

$$xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=y\frac{\partial u}{\partial y}(\cos 2u-1) \quad \text{—————— (3)}$$

Adding equation (2) and (3), we get

$$\left(x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}\right)+\left(xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}\right)=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)(\cos 2u-1)$$

Add.

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\sin 2u(\cos 2u-1)$$

$$=\frac{1}{2}\sin 2u(\cancel{1}-2\sin^2 u-\cancel{1})$$

$$=\frac{1}{2}\sin 2u(-2\sin^2 u)$$

$= -\sin 2u \sin^2 u$

Hence $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u.$

Proved

H.W If $u = \operatorname{cosec}^{-1}\left(\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}\right)^{\frac{1}{2}}$, Prove that

$x\frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u)$